1. Find the sum S for the following series. Hint: you can decompose the fraction into two fractions using use partial fractions. The series can then be calculated as a telescoping series.

$$S = \sum_{n=1}^{\infty} \frac{4}{4n^2 - 1}$$

From the series in Unit 5, we follow three steps: 1) find a general term for partial sum S_n, 2) take the limit of S_n for $n \to \infty$, 3) if the limit exist, the series is convergent and it is equal to the sum.
By decomposing the fraction into two fractions, using partial fractions (note, denominator is difference of two squares):

$$\frac{4}{4n^2 - 1} = \frac{4}{(2n-1)(2n+1)}$$

Decompose to two fractions:

$$\frac{4}{(2n-1)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n+1)} \to 4 = A(2n+1) + B(2n-1)$$

So,

$$\text{For } n = -1/2 : -2B = 4 \to B = -2.$$
$$\text{For } n = 1/2 : 2A = 4 \to A = 2.$$

Therefore:

$$\sum_{n=1}^{\infty} \frac{4}{4n^2 - 1} = \sum_{n=1}^{\infty} (\frac{2}{2n-1} - \frac{2}{2n+1})$$

Obtain S_n:

$$S_n = (\frac{2}{1} - \frac{2}{3}) + (\frac{2}{3} - \frac{2}{5}) + (\frac{2}{5} - \frac{2}{7}) + ... + (\frac{2}{2n-3} - \frac{2}{2n-1}) + (\frac{2}{2n-1} - \frac{2}{2n+1})$$

Simplifying S_n, we get:

$$S_n = (2 - \frac{2}{2n+1})$$

The last step is to take the limit:

$$\lim_{n \to \infty} (2 - \frac{2}{2n+1}) = 2$$

So the series converges to $S = 2$.

2. Prove that $f(x) = \sqrt{16 - x^2}$ is continuous on [-4, 4].
From continuity on intervals in Unit 6, we need to show three conditions are held: 1) If $f(x)$ is continuous for all $c \in (-4, 4)$, 2) If $\lim_{n \to -4^+} f(x) = f(-4)$, and 3) If $\lim_{n \to 4^-} f(x) = f(4)$.
First condition is held because for every c on (-4, 4):

$$\lim_{x \to c} \sqrt{16 - x^2} = \sqrt{16 - c^2} = f(c)$$

Second and third conditions are also held:

$$\lim_{x \to -4^+} \sqrt{16 - x^2} = \sqrt{16 - (-4)^2} = 0 = f(-4)$$
$$\lim_{x \to 4^-} \sqrt{16 - x^2} = \sqrt{16 - 4^2} = 0 = f(4)$$

So, because all conditions are true, then $f(x)$ is continuous on [-4, 4].

3. Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the interval [-2, 0].
From Unit 7, we need to follow two steps: 1) find the critical points and evaluate the function. 2) Evaluate $f(x)$ at the end points. The largest and smallest values of the function will be global max and min, respectively.
To find the critical points:

$$f'(x) = 0 \rightarrow f'(x) = 6x^2 - 6x - 12 = 0 \rightarrow f'(x) = 6(x^2 - x - 2) = 0 \rightarrow f'(x) = (x-2)(x+1) = 0$$

$$(x-2) = 0 \rightarrow x = 2 \text{ (not in the interval)}$$

$$(x+1) = 0 \rightarrow x = -1$$

Evaluate the function:

$$f(-1) = 8$$

$$f(-2) = -3$$

$$f(0) = 1$$

Global minimum is $f(-2) = -3$. Global maximum is $f(-1) = 8$.

4. Consider

$$f(x) = \frac{1}{1-x}$$

(a) Find the Maclaurin series of $f(x)$.
from Taylor and Maclaurin series in Unit 8, we follow two steps: i) fiend the derivative of function, and 2) evaluate derivatives at $x = 0$.
Maclaurin series of function $f(x)$ is defined:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We find the derivatives:

$$f^0(x) = \frac{1}{1-x} \rightarrow f^0(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \rightarrow f'(0) = 1$$

$$f''(x) = \frac{2(1-x)}{(1-x)^4} \rightarrow f''(0) = 2$$

$$f'''(x) = \frac{-2(1-x)^4 + 8(1-x)^4}{(1-x)^8} \rightarrow f'''(0) = 6$$

$$\ldots$$

$$f^{(n)}(0) = n!$$

So, we have:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

(b) Find the convergence interval for the series obtained from part (a). Use the ratio test.
Since this is a Geometric series, we use ratio test to obtain the convergence interval:

$$r = \lim_{n\to\infty} |\frac{x^{n+1}}{x^n}| = |x|$$

So $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $-1 < x < 1$.